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CONJUGACY CLASSES OF MAXIMAL NILPOTENT SUBGROUPS

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ABSTRACT

It is shown that in an arbitrary finite group G, any two maximal nilpotent subgroups of G whose intersection contains its own centralizer in G, are conjugate in G.

NOTATION. Let G be a group.

$\mathrm{Max}_N(G) = \{V : V \leqq G,\ V$ is nilpotent, and $V < W \leqq G$ implies that W is not nilpotent$\}$;

$B_2(G) = \{B : B \leqq G,\ B$ is nilpotent of class at most 2$\}$;

$e_2(G) = \max\{|B| : B \in B_2(G)\}$;

$A_2(G) = \{B : B \in B_2(G)$ and $|B| = e_2(G)\}$.

If G is nilpotent, then G_p will denote the Sylow p-subgroup of G, and $G_{p'}$ its p-complement in G.

PROPOSITION 1. *Let G be a group, $C_G(B) \leqq B \lhd G$, $X \in \mathrm{Max}_N(G)$, $X \geqq B$, and F the Fitting subgroup of G. Then X contains F.*

COROLLARY 1. *Let G be a group, $C_G(B) \leqq B \lhd G$, $X, Y \in \mathrm{Max}_N(G)$, and $X \cap Y \geqq B$. Then X and Y are conjugate in G.*

PROOF. Let $p \neq q$ be primes. Then $[F_p, X_q] \leqq [C_G(B_{p'}), C_G(B_p)] \leqq C_G(B_{p'}) \cap C_G(B_p) = C_G(B) \leqq B$ as $B \lhd G$. Hence $F_p \leqq N_G(BX_q)$ and, since $BX_q \leqq X$, it follows that $F_p \leqq N_G(X_q)$. Consequently, $F_p \leqq C_G(X_q)$; therefore XF_p is nilpotent, and as $X \in \mathrm{Max}_N(G)$, $F_p \leqq X$, for all primes p. Hence $F \leqq X$, which proves the proposition. As $C_G(F) \leqq C_G(B) \leqq B \leqq F$, the main result of [4] applies, i.e., any two subgroups $X, Y \in \mathrm{Max}_N(G)$ which contain B — and by Proposition 1 therefore contain F — are conjugate in G. This proves the corollary.

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PROPOSITION 2. *Let G be a group, $C_G(B) \leqq B \leqq G$, $X, Y \in \mathrm{Max}_N(G)$, and $X \cap Y \geqq B$. Then X and Y are conjugate in G.*

PROOF (P. Förster). Let G be a counterexample to Proposition 2 of minimal order and suppose that X and Y are not conjugate in G. Without loss of generality we may assume that $|X \cap Y| = \max\{|X^* \cap Y^*| : X^*, Y^* \in \mathrm{Max}_N(G), X^* \cap Y^* \geqq B$, and X^* and Y^* are not conjugate in $G\}$. We put $D = X \cap Y$, $N = N_G(D)$, $X_0 = N_X(D)$, $Y_0 = N_Y(D)$. If $D = X_0$, then the nilpotency of X implies $X = D \leqq Y$, hence $X = Y$, a contradiction. Therefore $D < X_0$ and likewise $D < Y_0$. If $N = G$, then $D \lhd G$, but $C_G(D) \leqq C_G(B) \leqq B \leqq D$ so that by the corollary of Proposition 1, X and Y would be conjugate in G, again a contradiction. Therefore $N < G$. We choose $X_1, Y_1 \in \mathrm{Max}_N(N)$ such that $X_0 \leqq X_1$, $Y_0 \leqq Y_1$. The choice of G ensures that there exists $n \in N$ such that $X_1 = Y_1^n$. Let $X_2 \in \mathrm{Max}_N(G)$ such that $X_1 \leqq X_2$. Then $Y^n \cap X_2 \geqq Y_0^n > D^n = D = X \cap Y \geqq B$, in particular $|Y^n \cap X_2| > |X \cap Y|$ and $Y^n \cap X_2 \geqq B$. Therefore $Y^{ng} = X_2$ for some $g \in G$. But $Y^{ng} \cap X \geqq X_0 > D = X \cap Y \geqq B$, so again $|Y^{ng} \cap X| > |X \cap Y|$ and $Y^{ng} \cap X \geqq B$ implies that Y^{ng} and X are conjugate in G. Hence X and Y are conjugate in G. This proves Proposition 2.

As an application of Proposition 2, we give a short proof of a result credited to Bender which can be found in [1]. Proposition 2 permits us to use the following result by Glauberman ([3]) more efficiently than in [1]: Let G be a group, $B \in A_2(G)$, $X \leqq G$, X nilpotent, and $B \leqq N_G(X)$. Then BX is nilpotent.

PROPOSITION 3. *Let G be a group, F its Fitting subgroup, $C_G(F) \leqq F$, and $X \in \mathrm{Max}_N(G)$. Then $X \geqq F$ if and only if there exists $B \in A_2(G)$ such that $X \geqq B$.*

PROOF. We note that if $B \in A_2(G)$, then $C_G(B) \leqq B$. Suppose $X \geqq B$ for some $B \in A_2(G)$. By [3], BF is nilpotent, therefore $BF \leqq Y$ for some $Y \in \mathrm{Max}_N(G)$. Hence $X \cap Y \geqq B$, so by Proposition 2, $Y^g = X$ for some $g \in G$. Therefore $X \geqq F$.

Conversely (see [1]), if $X \geqq F$ and $B \in A_2(G)$, BF is again nilpotent by [3], so $BF \leqq Y$ for some $Y \in \mathrm{Max}_N(G)$. Hence $X \cap Y \geqq F$, and [4] (or our Proposition 2) implies $Y^g = X$ for some $g \in G$. Therefore $X \geqq B^g$ with $B^g \in A_2(G)$.

Bialostocki conjectured in [2] that if G is a group, $X, Y \in \mathrm{Max}_N(G)$, $B_X, B_Y \in A_2(G)$ and $B_X \leqq X$, $B_Y \leqq Y$, then X and Y are conjugate in G. We state another conjecture which implies Bialostocki's conjecture.

CONJECTURE. If G is a group, $B, B^* \in A_2(G)$, then there exists $g \in G$ such that $\langle B^*, B^g \rangle$ is nilpotent.

Suppose this conjecture is true, then there exists $g \in G$ such that $\langle B_X, B_Y^g \rangle$ is nilpotent. Let $Z \in \mathrm{Max}_N(G)$ with $Z \geqq \langle B_X, B_Y^g \rangle$, then $Z \cap X \geqq B_X$, $Z \cap Y^g \geqq B_Y^g$, hence by Proposition 2, Z is a conjugate of X and of Y^g, so Bialostocki's conjecture would be true.

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